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Differential Forms

Differential forms are one of the most useful concepts from advanced mathematics. Like many advanced math concepts, you have secretly already been using them before. They provide one method for constructing coordinate invariant expressions, simplify certain calculations (e.g. curvature tensors) and play a central role in differential topology, i.e. De Rham cohomology.

To begin, a differential p-form $A^{(p)}$ is simply a $(0,p)$ tensor that is completely antisymmetric. In terms of components:

In 4D:

0-form ϕ (No indices) — 0 components

1-form A_μ (dual vector) A_0, A_1, A_2, A_3 — 4 components

2-form $B_{\mu\nu}$ where $B_{\nu\mu} = -B_{\mu\nu}$ $\quad B_{\mu\mu}=0$ $\quad B_{01}, B_{02}, B_{03}, B_{12}, B_{13}, B_{23}$ — 6 components

3-form $C_{\mu\nu\lambda}$ where $\underbrace{C_{\mu\nu\lambda} = C_{\lambda\mu\nu} = C_{\nu\lambda\mu}}_{\text{even \# of index swaps}} = \underbrace{-C_{\nu\mu\lambda} = -C_{\lambda\nu\mu} = -C_{\mu\lambda\nu}}_{\text{odd \# of index swaps}}$

$C_{\mu\mu\lambda} = 0$ $\quad C_{012}, C_{013}, C_{023}, C_{123}$ — $\underbrace{\text{4 components}}$ actually has fewer components than $B_{\mu\nu}$!!

For a p-form in D dimensions we have $\binom{D}{p} = \frac{D!}{p!(D-p)!}$ independent components.

Note: If $p = D$ then we have 1 component (recall $0! = 1$).
Furthermore $p \leq D$. When $p = D$ this is called the "top" form.

At the moment things might not seem so special, but the interesting things arise when we consider

- Products of forms
- Derivatives of forms
- Integrals of forms

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Wedge Product

We can "multiply" two forms and get another form if we are careful to preserve antisymmetry.

$A^{(p)} \wedge B^{(q)} = C^{(p+q)}$ in terms of components $C_{\mu_1 \cdots \mu_{p+q}} = (A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}$

Example: $p = q = 1$ $\quad (A \wedge B)_{\mu\nu} = \frac{2!}{1!\,1!} A_{[\mu} B_{\nu]} = 2 \cdot \frac{1}{2} (A_\mu B_\nu - A_\nu B_\mu) = A_\mu B_\nu - A_\nu B_\mu$

Example: $p=1$ $q=2$ $\quad (A \wedge B)_{\mu\nu\lambda} = \frac{3!}{1!\,2!} A_{[\mu} B_{\nu\lambda]} = 3 \cdot \frac{1}{6} (A_\mu B_{\nu\lambda} + A_\lambda B_{\mu\nu} + A_\nu B_{\lambda\mu} - A_\mu B_{\lambda\nu} - A_\nu B_{\mu\lambda} - A_\lambda B_{\nu\mu})$

$$= \frac{1}{2} (A_\mu B_{\nu\lambda} + A_\lambda B_{\mu\nu} + A_\nu B_{\lambda\mu} - A_\mu B_{\lambda\nu} - A_\nu B_{\mu\lambda} - A_\lambda B_{\nu\mu})$$

What might not be obvious is that: $A \wedge B = (-1)^{pq} B \wedge A$

Example: $p = q = 1$ $\quad (B \wedge A)_{\mu\nu} = B_\mu A_\nu - B_\nu A_\mu = -(A_\mu B_\nu - A_\nu B_\mu) = -(A \wedge B)_{\mu\nu}$

Example: $p=1$ $q=2$ $\quad (B \wedge A)_{\mu\nu\lambda} = \frac{1}{2} (B_{\mu\nu} A_\lambda + B_{\lambda\mu} A_\nu + B_{\nu\lambda} A_\mu - B_{\lambda\nu} A_\mu - B_{\mu\lambda} A_\nu - B_{\nu\mu} A_\lambda)$

$$= \frac{1}{2} (A_\mu B_{\nu\lambda} + A_\lambda B_{\mu\nu} + A_\nu B_{\lambda\mu} - A_\mu B_{\lambda\nu} - A_\nu B_{\mu\lambda} - A_\lambda B_{\nu\mu})$$

$$= (A \wedge B)_{\mu\nu\lambda}$$

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Exterior derivative

Forms (their components at least) can vary over space and time, so when we write $B_{\mu\nu}$ we really mean $B_{\mu\nu}(x^\lambda)$.

It would of course be useful to describe how forms vary, so a derivative would be useful. We could just use the usual partial derivative $\partial_\lambda \equiv \frac{\partial}{\partial x^\lambda}$, but the object that results is not particularly nice. On the other hand if we are careful we can make a derivative such that the derivative of a form gives another form!

Consider $A^{(p)}$ then dA is a $(p+1)$-form w/ components $(dA)_{\mu_1 \cdots \mu_{p+1}} = (p+1)!\, \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}$

This exterior derivative has (at least) 3 nice properties:

a) It satisfies a modified Leibniz (product) rule.

For $A^{(p)}$ and $B^{(q)}$ we have $d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB$

b) You might think that on a curved space, or in curvilinear coordinates we would need to use $\partial_\mu \to D_\mu = \partial_\mu + \Gamma^{\cdot}_{\cdot\cdot}$ to get back a true tensor (form).
But it turns out that due to the antisymmetrization, even if you include Christoffel connections, they end up cancelling out.
This means that we have a meaningful tensorial derivative w/out requiring a metric!
So these derivatives are defined on and only depend on the topological aspects of the space(time)!

c) It "squares" to zero.

$d(dA) = 0$ since we would have to antisymmetrize the indices on each ∂, but partial derivatives commute!

This feature is one of the keys to how exterior calculus of differential forms leads to topological invariants.

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Integration

Once you have a derivative, the natural question is "how do we integrate?"

Again there is magic here.

So far we have been looking only at the components of forms. Here it will be useful to remember that the components arise when we decompose the form onto a basis.

What is the basis?

$A^{(p)} = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p}$ where the differentials anticommute, i.e. $dx^\mu dx^\nu = -dx^\nu dx^\mu$

Why? Well consider $A^{(2)} = \frac{1}{2} A_{\mu\nu} dx^\mu dx^\nu$ but since μ,ν are dummy we can rename them $\mu \leftrightarrow \nu$

$= \frac{1}{2} \underbrace{A_{\nu\mu}}_{} \underbrace{dx^\nu dx^\mu}_{}$

but $= -A_{\mu\nu}$ so $= -dx^\mu dx^\nu$

In fact we could consider this as $dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$.

What is so nice about a set of anticommuting differentials?

Consider $dxdy$ and transform to $x'(x,y)$ $y'(x,y)$. Then:

$$dxdy \longrightarrow dx'dy' = \left(\frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy\right)\left(\frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy\right) = \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x}dxdx + \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial y}dydy + \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y}dxdy + \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}dydx$$

However, we know how $dxdy$ should transform... w/ the Jacobian, i.e. $dx'dy' = \left(\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}\right)dxdy$

Which is exactly what we get if $dxdy = -dydx \Rightarrow dxdx = dydy = 0$

So at the end of the day, the basis of a p-form is actually an integration measure over a p-dimensional (oriented) volume!

This means that an expression like $\int_{\Sigma_p} A^{(p)}$ is perfectly well defined and coordinate invariant.

The physical significance of this is that there is a "natural" coupling between p-form fields $A^{(p)}$ and the p-dimensional "world-surfaces" swept out by $(p-1)$-dimensional objects.

Example: $p=1$ $\int_{\Sigma_1} A^{(1)}$ is the natural coupling of a 1-form A_μ to a particle's worldline.

Example: $p=2$ $\int_{\Sigma_2} B^{(2)}$ is the natural coupling of a 2-form $B_{\mu\nu}$ to a string's worldsheet.

In fact $\int_{\Sigma_p} B^{(p)}$ is the natural coupling of a p-form to a $D(p-1)$-brane's worldvolume.